

Analog of Electroweak Model for Contracted Gauge Group

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Abstract

The limiting case of the bosonic part of the standard Electroweak Model, which correspond to the contracted gauge group $SU(2; j) \times U(1)$ is discussed. The Higgs boson, Z-boson and electromagnetic fields become an external ones with respect to the W-bosons fields. The W-bosons fields do not effect on these external fields. The masses of the all particles of the Electroweak Model remain the same under contraction.

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1 Introduction

The Standard Electroweak Model based on gauge group $SU(2) \times U(1)$ gives a good description of electroweak processes. One of its important ingredient is the simple group $SU(2)$. On the other hand, in physics it is well known group contraction [1], which transforms, for example, a simple or semisimple group to a non-semisimple one. For better understanding of a physical system it is useful to investigate its limits for limiting values of their physical parameters.

In the present paper we construct the analog of the bosonic part of the standard Electroweak Model as the gauge theory with the contracted non-semisimple group $SU(2; j) \times U(1)$. The gauge theories for non-semisimple groups which Lie algebras admit invariant non-degenerate metrics was considered in [2], [3]. The construction given in [4] is based on an observation: the underlying group of the standard Electroweak Model can be represented as a semidirect product of $U(1)$ and $SU(2)$.

Most often the fundamental representation of the contracted special unitary group $SU(2; j)$ is represented by the triangular matrix

$$u = \begin{pmatrix} a & 0 \\ b & \bar{a} \end{pmatrix}, \quad \det u = a\bar{a} = 1$$

with complex matrix elements $a, b \in \mathbf{C}$. But there is another possibility [5]. Instead of matrix with complex elements we can use matrix with nilpotent elements in the form

$$u(\iota) = \begin{pmatrix} \alpha & \iota\beta \\ -\iota\bar{\beta} & \bar{\alpha} \end{pmatrix}, \quad \det u(\iota) = |\alpha|^2 = 1, \quad \alpha, \beta \in \mathbf{C},$$

where nilpotent unit $\iota \neq 0$, but $\iota^2 = 0$. These two possibilities correspond to the nonsymmetric appearance of contraction parameter j in the first case, when the matrix element u_{12} is multiplied by $j^2 = \iota^2 = 0$ and the symmetric one, when matrix elements u_{12}, u_{21} are multiplied by $j = \iota$.

In Sec. II, we recall the definition and properties of the contracted group $SU(2; j)$. In Sec. III, we step by step modify the main points of the Electroweak Model for the gauge group $SU(2; j) \times U(1)$. The use of the representation with the symmetric arranged contraction parameter enables us to find transformation properties (20) of gauge fields. After that the Lagragian of the contracted model can be very easy obtained from the standard one.

Two interpretations of the contracted model are discussed. In the first one the Higgs boson, Z-boson and electromagnetic fields can be regarded as an external fields with respect to the W-bosons fields and the last ones do not effect on these external fields. The masses of the all particles of the Electroweak Model do not changed under contraction.

2 Contracted Special Unitary group $SU(2; j)$

Let us regard two dimensional complex fibered vector space $\Phi_2(j)$ with one dimensional base $\{\phi_2\}$ and one dimensional fiber $\{\phi_1\}$ [5]. This space has two hermitian forms: first in the base $\bar{\phi}_2\phi_2 = |\phi_2|^2$ and second in the fiber $\bar{\phi}_1\phi_1 = |\phi_1|^2$, where bar denotes complex conjugation. Both forms can be written by one formula

$$\phi^\dagger\phi(j) = j^2|\phi_1|^2 + |\phi_2|^2, \quad (1)$$

where $\phi^\dagger = (j\bar{\phi}_1, \bar{\phi}_2)$, parameter $j = 1, \iota$ and ι is nilpotent unit $\iota^2 = 0$. We shall demand that the following heuristic rules be fulfilled: for a real or complex a the expression $\frac{a}{\iota}$ is defined only for $a = 0$, however $\frac{\iota}{\iota} = 1$. The usual complex vector space Φ_2 is obtained for $j = 1$. Inverse correspondence is given by

$$|\phi_2|^2 = \phi^\dagger\phi(j)|_{j=\iota}; \quad |\phi_1|^2 = \frac{1}{j^2}\phi^\dagger\phi(j)|_{j=\iota}, \text{ for } \phi_2 = 0. \quad (2)$$

The special unitary group $SU(2; j)$ is defined as a transformation group of $\Phi_2(j)$ which keep invariant the hermitian form (1), i.e.

$$\begin{aligned}\phi'(j) &= \begin{pmatrix} j\phi'_1 \\ \phi'_2 \end{pmatrix} = \begin{pmatrix} \alpha & j\beta \\ -j\bar{\beta} & \bar{\alpha} \end{pmatrix} \begin{pmatrix} j\phi_1 \\ \phi_2 \end{pmatrix} = u(j)\phi(j), \\ \det u(j) &= |\alpha|^2 + j^2|\beta|^2 = 1.\end{aligned}\quad (3)$$

The fundamental representation of the one-parameter subgroups of $SU(2; j)$ are easily obtained

$$\omega_1(\alpha_1; j) = e^{i\alpha_1 T_1(j)} = \begin{pmatrix} \cos \frac{j\alpha_1}{2} & i \sin \frac{j\alpha_1}{2} \\ i \sin \frac{j\alpha_1}{2} & \cos \frac{j\alpha_1}{2} \end{pmatrix}, \quad (4)$$

$$\omega_2(\alpha_2; j) = e^{i\alpha_2 T_2(j)} = \begin{pmatrix} \cos \frac{j\alpha_2}{2} & \sin \frac{j\alpha_2}{2} \\ -\sin \frac{j\alpha_2}{2} & \cos \frac{j\alpha_2}{2} \end{pmatrix}, \quad (5)$$

$$\omega_3(\alpha_3; j) = e^{i\alpha_3 T_3(j)} = \begin{pmatrix} e^{i\frac{\alpha_3}{2}} & 0 \\ 0 & e^{-i\frac{\alpha_3}{2}} \end{pmatrix}. \quad (6)$$

The corresponding generators

$$T_1(j) = \frac{j}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad T_2(j) = \frac{j}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad T_3(j) = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (7)$$

with commutation relations

$$\begin{aligned}[T_1(j), T_2(j)] &= -ij^2 T_3(j), \quad [T_3(j), T_1(j)] = -iT_2(j), \\ [T_2(j), T_3(j)] &= -iT_1(j),\end{aligned}\quad (8)$$

form the Lie algebra $su(2; j)$.

There are two more or less equivalent way of group contraction. We can put the contraction parameter equal to the nilpotent unit $j = \iota$ or tend it to zero $j \rightarrow 0$. Sometimes it is convenient to use mathematical approach, sometimes physical one. For example the matrix $u(j)$ in (3) has nilpotent non-diagonal elements for $j = \iota$, whereas for $j \rightarrow 0$ they are formally equal to zero. Nevertheless both approaches lead to the same final results.

Let us describe the contracted group $SU(2; \iota)$ in detail. For $j = \iota$ it follows from (3) that $\det u(\iota) = |\alpha|^2 = 1$, i.e. $\alpha = e^{i\varphi}$, therefore

$$u(\iota) = \begin{pmatrix} e^{i\varphi} & \iota\beta \\ -\iota\bar{\beta} & e^{-i\varphi} \end{pmatrix}, \quad \beta = \beta_1 + i\beta_2 \in \mathbf{C}. \quad (9)$$

Functions of nilpotent arguments are defined by their Taylor expansion, in particular, $\cos \iota x = 1$, $\sin \iota x = \iota x$. Then one-parameter subgroups of $SU(2; \iota)$ take the form

$$\begin{aligned}\omega_1(\alpha_1; \iota) &= e^{i\alpha_1 T_1(\iota)} = \begin{pmatrix} 1 & \iota i \frac{\alpha_1}{2} \\ \iota i \frac{\alpha_1}{2} & 1 \end{pmatrix}, \\ \omega_2(\alpha_2; \iota) &= e^{i\alpha_2 T_2(\iota)} = \begin{pmatrix} 1 & \iota \frac{\alpha_2}{2} \\ -\iota \frac{\alpha_2}{2} & 1 \end{pmatrix}. \end{aligned}\quad (10)$$

The third subgroup does not change and is given by (6). The simple group $SU(2)$ is contracted to the non-semisimple group $SU(2; \iota)$, which is isomorphic to the real Euclid group $E(2)$. First two generators of the Lie algebra $su(2; \iota)$ commute $[T_1(\iota), T_2(\iota)] = 0$ and the rest commutators are given by (8). For the general element $T(\iota) = a_1 T_1(\iota) + a_2 T_2(\iota) + a_3 T_3(\iota)$ of $su(2; \iota)$ the corresponding group element of $SU(2; \iota)$ is as follows

$$u(\iota) = e^{iT(\iota)} = \begin{pmatrix} e^{i\frac{a_3}{2}} & \iota \frac{a}{a_3} \sin \frac{a_3}{2} \\ -\iota \frac{a}{a_3} \sin \frac{a_3}{2} & e^{-i\frac{a_3}{2}} \end{pmatrix}, \quad a = a_2 + ia_1. \quad (11)$$

The actions of the unitary group $U(1)$ and the electromagnetic subgroup $U(1)_{em}$ in the fibered space $\Phi_2(\iota)$ are given by the same matrices as on the space Φ_2 , namely

$$\omega(\beta) = e^{i\beta Y} = \begin{pmatrix} e^{i\frac{\beta}{2}} & 0 \\ 0 & e^{i\frac{\beta}{2}} \end{pmatrix}, \quad \omega_{em}(\gamma) = e^{i\gamma Q} = \begin{pmatrix} 1 + i\gamma & 0 \\ 0 & 1 \end{pmatrix}. \quad (12)$$

Representations of groups $SU(2; \iota), U(1), U(1)_{em}$ are linear ones, that is they are realised by linear operators in the fibered space $\Phi_2(\iota)$.

3 Electroweak Model for $SU(2; j) \times U(1)$ gauge group

We shall follow the book [6] in description of standard Electroweak Model. The bosonic sector of Electroweak Model is $SU(2; j) \times U(1)$ gauge theory in the space $\Phi_2(j)$ of fundamental representation of $SU(2; j)$. The bosonic Lagrangian is given by the sum

$$L_B(j) = L_A(j) + L_\phi(j), \quad (13)$$

where

$$\begin{aligned} L_A(j) &= \frac{1}{8g^2} \text{Tr}(F_{\mu\nu}(j))^2 - \frac{1}{4}(B_{\mu\nu})^2 = \\ &= -\frac{1}{4}[j^2(F_{\mu\nu}^1)^2 + j^2(F_{\mu\nu}^2)^2 + (F_{\mu\nu}^3)^2] - \frac{1}{4}(B_{\mu\nu})^2 \end{aligned} \quad (14)$$

is the gauge field Lagrangian for $SU(2; j) \times U(1)$ group and

$$L_\phi(j) = \frac{1}{2}(D_\mu\phi(j))^\dagger D_\mu\phi(j) - V(\phi(j)) \quad (15)$$

is the matter field Lagrangian (summation on the repeating Greek indexes is always understood). Here $\phi(j) = \begin{pmatrix} j\phi_1 \\ \phi_2 \end{pmatrix} \in \Phi_2(j)$, D_μ are the covariant derivatives

$$D_\mu\phi(j) = \partial_\mu\phi(j) - ig \left(\sum_{k=1}^3 T_k(j) A_\mu^k \right) \phi(j) - ig' Y B_\mu\phi(j), \quad (16)$$

where $T_k(j)$ are given by (7) and $Y = \frac{1}{2}\mathbf{1}$ is generator of $U(1)$. The gauge fields

$$A_\mu(x; j) = -ig \sum_{k=1}^3 T_k(j) A_\mu^k(x), \quad B_\mu(x) = -ig' B_\mu(x)$$

take their values in Lie algebras $su(2; j)$, $u(1)$ respectively, and the stress tensors are

$$F_{\mu\nu}(x; j) = \mathcal{F}_{\mu\nu}(x; j) + [A_\mu(x; j), A_\nu(x; j)], \quad B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu.$$

The potential $V(\phi(j))$ in (15) is introduced by hand in a special form (sombrero)

$$V(\phi(j)) = \frac{\lambda}{4} (\phi^\dagger(j)\phi(j) - v^2)^2, \quad (17)$$

where λ, v are constants.

The Lagrangian $L_B(j)$ (13) describe massless fields. To generate mass terms for the vector bosons without breaking the gauge invariance one uses the Higgs mechanism. One of $L_B(j)$ ground states

$$\phi^{vac} = \begin{pmatrix} 0 \\ v \end{pmatrix}, \quad A_\mu^k = B_\mu = 0$$

is taken as a vacuum state of the model, and small field excitations

$$\phi_1(x), \quad \phi_2(x) = v + \chi(x), \quad A_\mu^a(x), \quad B_\mu(x)$$

with respect to the vacuum are regarded. The matrix $Q = Y + T^3 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, which annihilates the ground state $Q\phi^{vac} = 0$, is the generator of the electromagnetic subgroup $U(1)_{em}$. The new fields

$$\begin{aligned} W_\mu^\pm &= \frac{1}{\sqrt{2}} (A_\mu^1 \mp i A_\mu^2), \quad Z_\mu = \frac{1}{\sqrt{g^2 + g'^2}} (g A_\mu^3 - g' B_\mu), \\ A_\mu &= \frac{1}{\sqrt{g^2 + g'^2}} (g' A_\mu^3 + g B_\mu) \end{aligned} \quad (18)$$

are introduced, where W_μ^\pm are complex $\bar{W}_\mu^- = W_\mu^+$ and Z_μ, A_μ are real.

For small fields the Lagrangian L_B (13) can be rewritten in the form

$$L_B(j) = L_0(j) + L_{int}(j),$$

where $L_0(j) = L_B^{(2)}(j)$ is the usual second order Lagrangian for free vector and scalar bosons, and higher order terms $L_{int}(j)$ are regarded as field interactions. The second order terms of the Lagrangian (13) are as follows

$$\begin{aligned} L_B^{(2)}(j) &= \frac{1}{2} (\partial_\mu \chi)^2 - \frac{1}{2} m_\chi^2 \chi^2 - \frac{1}{4} \mathcal{Z}_{\mu\nu} \mathcal{Z}_{\mu\nu} + \frac{1}{2} m_Z^2 Z_\mu Z_\mu - \frac{1}{4} \mathcal{F}_{\mu\nu} \mathcal{F}_{\mu\nu} \\ &\quad + j^2 \left\{ -\frac{1}{2} \mathcal{W}_{\mu\nu}^+ \mathcal{W}_{\mu\nu}^- + m_W^2 W_\mu^+ W_\mu^- \right\} = L_1^{(2)} + j^2 L_W^{(2)}, \end{aligned} \quad (19)$$

where $\mathcal{Z}_{\mu\nu} = \partial_\mu Z_\nu - \partial_\nu Z_\mu$, $\mathcal{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $\mathcal{W}_{\mu\nu}^\pm = \partial_\mu W_\nu^\pm - \partial_\nu W_\mu^\pm$.

For $j = 1$ Lagrangian (19) describe massive vector fields W_μ^\pm with identical mass $m_W = \frac{1}{2}gv$ (charged W -bosons), massless vector field A_μ , $m_A = 0$ (photon), massive vector field Z_μ , with the mass $m_Z = \frac{v}{2}\sqrt{g^2 + g'^2}$ (neutral Z -boson) and massive scalar field χ , $m_\chi = \sqrt{2\lambda}v$ (Higgs boson). W - and Z -bosons have been observed and have the masses $m_W = 80\text{GeV}$, $m_Z = 91\text{GeV}$. Higgs boson has not been experimentally verified up to now.

The fibered space $\Phi_2(j)$ can be obtained from Φ_2 by substituting $j\phi_1$ instead of ϕ_1 . Substitution $\phi_1 \rightarrow j\phi_1$ induces another ones for Lie algebra generators $T_1 \rightarrow jT_1$, $T_2 \rightarrow jT_2$, $T_3 \rightarrow T_3$. As far as the gauge fields take their values in Lie algebra, we can substitute gauge fields instead of transformation of generators, namely

$$A_\mu^1 \rightarrow jA_\mu^1, \quad A_\mu^2 \rightarrow jA_\mu^2, \quad A_\mu^3 \rightarrow A_\mu^3, \quad B_\mu \rightarrow B_\mu. \quad (20)$$

For the gauge fields (18) and the corresponding stress tensors these substitutions are as follows

$$\begin{aligned} W_\mu^\pm &\rightarrow jW_\mu^\pm, \quad Z_\mu \rightarrow Z_\mu, \quad A_\mu \rightarrow A_\mu, \\ \mathcal{W}_{\mu\nu}^+ &\rightarrow j\mathcal{W}_{\mu\nu}^+, \quad \mathcal{Z}_{\mu\nu} \rightarrow \mathcal{Z}_{\mu\nu}, \quad \mathcal{F}_{\mu\nu} \rightarrow \mathcal{F}_{\mu\nu}. \end{aligned} \quad (21)$$

The matter field ϕ_2 does not transform as well as its small part χ . These substitutions in the standard Lagrangian $L_B^{(2)}$ of Electroweak Model give rise to (19).

Let contraction parameter tends to zero $j^2 \rightarrow 0$, then the contribution of W-bosons fields to the Lagrangian (19) will be small in comparison with Higgs boson, Z-boson and electromagnetic fields. In other words the limit model includes only three last fields and charded W-bosons fields does not effect on these fields. The part $L_W^{(2)}$ form a new Lagrangian for W-bosons fields. The appearance of two Lagrangians is in correspondence with two hermitian forms of fibered space $\Phi_2(\iota)$, which are invariant under the action of contracted gauge group $SU(2; \iota)$. Higgs boson, Z-boson and electromagnetic fields are appeared as external fields with respect to the W-bosons fields.

In mathematical language the field space $\{\chi, A_\mu, Z_\mu, W_\mu^\pm\}$ is fibered after contraction to the base $\{\chi, A_\mu, Z_\mu\}$ and the fiber $\{W_\mu^\pm\}$. (In order to avoid terminological misunderstanding let us stress that we have in view locally trivial fibering, which is defined by the projection $pr : \{\chi, A_\mu, Z_\mu, W_\mu^\pm\} \rightarrow \{\chi, A_\mu, Z_\mu\}$ in the field space. This fibering has nothing to do with the principal bundle.) Then $L_1^{(2)}$ in (19) presents Lagrangian in the base and $L_W^{(2)}$ is Lagrangian in the fiber. In general, properties of a fiber are dependent on a points of a base and not the reverse. In this sense fields in the base are external with respect to fields in the fiber.

To demonstrate this statement let us transform the formula (6.59) in [6] for interactions of the electromagnetic field with all other fields by substitution (21)

$$L_{int}^{em} = -\frac{1}{4}\mathcal{H}_{\mu\nu}^2 - j^2\frac{1}{2}\left\{|\mathcal{W}_{\mu\nu}^-|^2 + ig\mathcal{H}_{\mu\nu}\mathcal{P}_{\mu\nu} + j^2\frac{g^2}{2}\mathcal{P}_{\mu\nu}^2\right\}, \quad (22)$$

where

$$\mathcal{H}_{\mu\nu} = \mathcal{F}_{\mu\nu} \sin \theta + \mathcal{Z}_{\mu\nu} \cos \theta, \quad \mathcal{P}_{\mu\nu} = \mathcal{W}_\mu^- \mathcal{W}_\nu^+ - \mathcal{W}_\nu^- \mathcal{W}_\mu^+. \quad (23)$$

The first term in (22)

$$-\frac{1}{4}\mathcal{H}_{\mu\nu}^2 = -\frac{1}{4}\left\{\mathcal{F}_{\mu\nu}^2 \sin^2 \theta + \mathcal{F}_{\mu\nu}\mathcal{Z}_{\mu\nu} \sin 2\theta + \mathcal{Z}_{\mu\nu}^2 \cos^2 \theta\right\} \quad (24)$$

describes self-action of the electromagnetic field, interaction with Z-boson field and self-action of Z-boson field (all in the base). The last term in (22) disappears as having fourth order in $j \rightarrow 0$ and we obtain the following Lagrangian in the fiber

$$L_f = -\frac{1}{2}\mathcal{W}_{\mu\nu}^+\mathcal{W}_{\mu\nu}^- + m_W^2 W_\mu^+ W_\mu^- - \frac{1}{2}\left\{|\mathcal{W}_{\mu\nu}^-|^2 + ig\mathcal{H}_{\mu\nu}\mathcal{P}_{\mu\nu}\right\}, \quad (25)$$

which describes self-actions and interactions of W-bosons fields as well as their interactions with external electromagnetic and Z-boson fields. Let us note that interactions in contracted model are more simple as compared with the standard Electroweak Model due to nullification of some terms.

4 Conclusion

The suggested contracted model with the gauge group $SU(2; j) \times U(1)$, where $j = \iota$ or $j \rightarrow 0$, corresponds to the limiting case of the standard Electroweak Model. The masses of the all particles involved in the Electroweak Model remain the same under contraction, but interactions of the fields are changed in two aspects. Firstly all field interactions become simpler. Secondly inter-relations of the fields become more complicated. All fields are divided on two class: fields in the base (Higgs boson, Z-boson and electromagnetic fields) and fields in the fiber (W-bosons fields). The fields of the first class are external with respect to the last one. The Higgs boson, Z-boson and electromagnetic fields can interact with W-bosons fields, but in the limit W-bosons fields do not effect on these fields.

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